Stokes' Theorem

Consider a vector field $\mathbf{B}(\bar{r})$ where:

$$\mathbf{B}(\bar{r}) = \nabla x \mathbf{A}(\bar{r})$$

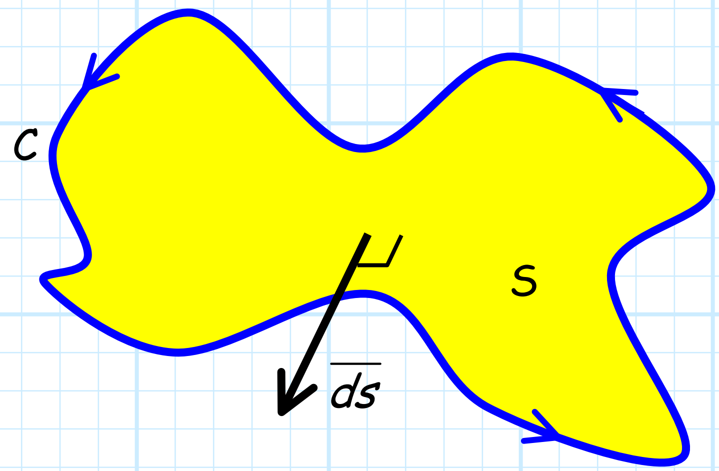
Say we wish to integrate this vector field over an **open** surface S:

$$\iint_S \mathbf{B}(\bar{r}) \cdot \overline{ds} = \iint_S \nabla x \mathbf{A}(\bar{r}) \cdot \overline{ds}$$

We can likewise evaluate this integral using **Stokes' Theorem**:

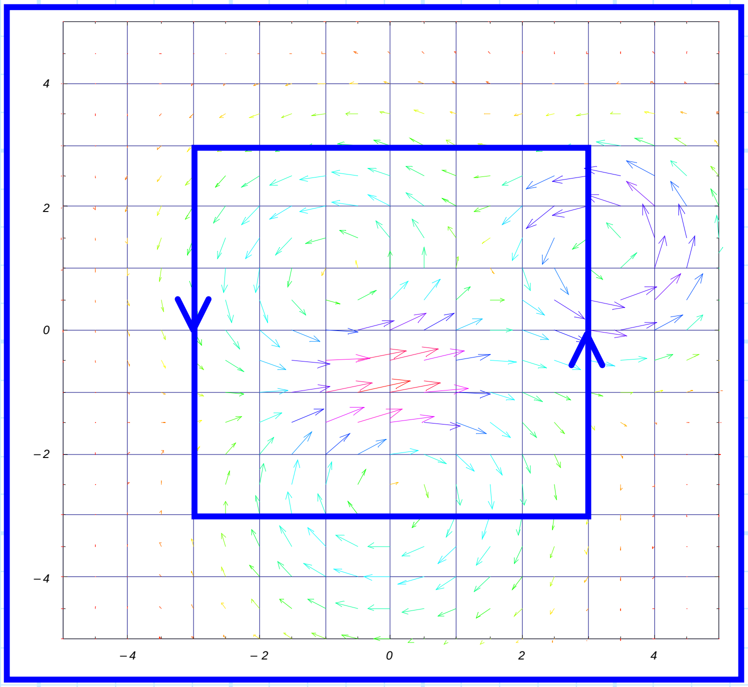
$$\iint_S \nabla x \mathbf{A}(\bar{r}) \cdot \overline{ds} = \oint_C \mathbf{A}(\bar{r}) \cdot \overline{d\ell}$$

In this case, the contour C is a **closed** contour that **surrounds** surface S. The direction of C is defined by $\overline{ds}$ and the **right - hand rule**. In other words C rotates **counter clockwise** around $\overline{ds}$. E.G.,



* Stokes' Theorem allows us to evaluate the **surface** integral of a curl as simply a **contour** integral !

* Stokes' Theorem states that the summation (i.e., integration) of the circulation at **every** point on a surface is simply the **total** "circulation" around the closed **contour** surrounding the surface.



In other words, if the vector field is **rotating counter-clockwise** around some point in the volume, it must simultaneously be **rotating clockwise** around adjacent points within the volume—the net effect is therefore **zero**!

Thus, the only values that make **any** difference in the **surface integral** is the rotation of the vector field around points that lie on the surrounding contour (i.e., the very edge of the surface S). These vectors are likewise rotating in the opposite direction around adjacent points—but these points do **not** lie on the surface (thus, they are **not** included in the integration). The net effect is therefore **non-zero**!

Note that if S is a **closed surface**, then there is **no** contour C that exists! In other words:

$$\oiint_S \nabla \times \mathbf{A}(\bar{r}) \cdot \overline{ds} = \oint_0 \mathbf{A}(\bar{r}) \cdot \overline{d\ell} = 0$$

Therefore, integrating the **curl of any vector field** over a **closed** surface **always** equals zero.